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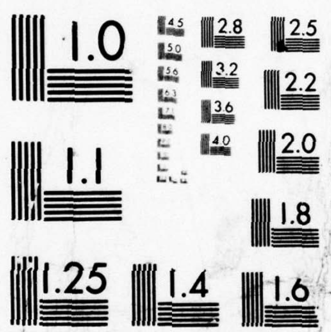

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VALUE THEORY WITHOUT EFFICIENCY

by

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Pradeep Dubey, Abraham Neyman [redacted] Robert James Weber

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VALUE THEORY WITHOUT EFFICIENCY*

by

Pradeep Dubey,[1] Abraham Neyman[2] and Robert James Weber[1]

0. Introduction

Recently attention has been focused on generalizations and analogues of the Shapley value that do not enjoy the efficiency, or Pareto optimality, property ([7], [9]). This has stemmed from the search for value functions that describe the prospects of playing different roles in a game (instead of describing fair division, in which case efficiency is a natural requirement). The purpose of this paper is to treat the subject from an axiomatic viewpoint, i.e., to characterize the class of operators that is obtained by omitting the efficiency axiom from the axioms defining the Shapley value. We consider both finite-player and nonatomic games. In the finite case, a complete solution is given; in the nonatomic case, a complete solution is given for the important space pNA .

1. The Finite Case

Let U be an infinite set, the universe of players. A game on U is a set function $v : 2^U \to R$ with $v(\emptyset) = 0$. We interpret the members of U as players and the members of 2^U as coalitions. A set $N \subset U$

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is a <u>support</u> of v if, for each $S \subset U$, $v(S) = v(S \cap N)$. A <u>finite game</u> is a game which has a finite support. We denote by G the vector space of all finite games, and by G^N the subspace of G consisting of games with support N. Let AG (respectively, AG^N) be the subspace of G (respectively, G^N) of additive games. (Note that for N finite, AG^N is isomorphic to R^N, the Euclidean space of dimension $|N|$ whose axes are indexed by the elements of N. For convenience we shall often use R^N for AG^N.)

Given a permutation θ of U (i.e., a 1-1 mapping from U onto itself) define the game θ^*v by $(\theta^*v)(S) = v(\theta S)$. Finally define v to be monotonic if $v(S) \geq v(T)$ whenever $S \supset T$.

A <u>semivalue</u> on G is a function $\Psi : G \to AG$ such that:

(1) Ψ is linear,

(2) $\Psi\theta^* = \theta^*\Psi$, for each permutation θ of U,

(3) if v is monotonic, then Ψv is monotonic,

(4) if $v \in AG$, then $\Psi v = v$.

These are the linearity, symmetry, monotonicity and projection axioms ([1], pp. 15-16). The projection axiom is an easy consequence of the more familiar dummy axiom, which says that if i is a dummy player in v (i.e., $v(S \cup i) = v(S) + v(i)$ whenever $i \notin S$) then $(\Psi v)(i) = v(i)$. (We conventionally omit the braces when indicating one-element sets.) The quantity $(\Psi v)(i)$, for $i \in U$, is a measure (according to Ψ) of the prospect of having role i in the game v.

Let ξ be a probability measure on $[0,1]$. For any $i \in U$ and any $v \in G$ with finite support N, define $\Psi_\xi v \in AG$ by

$$(1.1) \qquad (\Psi_\xi v)(i) = \sum_{S \subset N \setminus i} p_s^n [v(S \cup i) - v(S)] ,$$

where

$$p_s^n = \int_0^1 t^s(1-t)^{n-s-1}d\xi(t) \ .$$

(The symbols n and s generically denote the cardinalities of the sets N and S.) Note that the right-hand side of (1.1) is independent of the choice of N, so the definition makes sense.

We now come to our characterization of semivalues on G.

Theorem 1a. For each probability measure ξ on $[0,1]$, Ψ_ξ is a semivalue. Moreover, every semivalue on G is of this form, and the mapping $\xi \to \Psi_\xi$ is 1-1.

To prove this theorem we first characterize the semivalues on the vector space of games on a fixed finite-player set. This characterization has appeared elsewhere (see, for example, [9]). For the sake of completeness, we present an alternative derivation here. Then we proceed with two different proofs which shed light on Theorem 1a from different viewpoints. Let $N \subset U$ be a finite set. A semivalue on G^N is a function $\psi^N : G^N \to AG^N$ satisfying (1), (2^N), (3), (4), where (2^N) requires that $\psi^N\theta* = \theta*\psi^N$ for every N-preserving permutation θ of U.

Let $p^n = (p_0^n, \ldots, p_{n-1}^n)$ be a vector such that $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^n = 1$ and $p^n \geq 0$. Define $\psi_{p^n}^N : G^N \to AG$ by

$$(1.2) \qquad (\psi_{p^n}^N v)(i) = \sum_{S \subset N \setminus i} p_s^n [v(S \cup i) - v(S)]$$

for all $i \in N$ and $v \in G^N$.

<u>Lemma</u>. For each vector p^n , $\Psi^N_{p^n}$ is a semivalue on G^N . Moreover, every semivalue on G^N is of this form, and the mapping $p^n \to \Psi^N_{p^n}$ is 1-1.

<u>Proof</u>. It is straightforward to verify that each $\Psi^N_{p^n}$ is indeed a semivalue. Without loss of generality take $N = \{1, \ldots, n\}$, and let Ψ^N be a semivalue on G^N . Consider the vector space F of symmetric linear functions from G^N and AG^N . For any nonempty $S \subset N$, define the game $v_S \in G^N$ by $v_S(T) = 1$ if $S \subset T$, $v_S(T) = 0$ otherwise. It is well-known (see, for example, Appendix A of [1]) that $\{v_S : \emptyset \neq S \subset N\}$ is a basis for G^N ; therefore, every element $f \in F$ is uniquely determined by its values on the games in this basis. From the symmetry axiom (2), it is in fact sufficient to specify $f(v)$ for every $v \in \{v_{S(k)} : 1 \leq k \leq n\}$ where $S(k) = \{1, \ldots, k\}$. Hence the dimension of F is at most n .

For each $0 \leq k \leq n-1$ let $\Psi_{(k)} = \Psi^N_p$, as defined by (1.2) when $p_k = \binom{n-1}{k}^{-1}$ and $p_\ell = 0$ for all $\ell \neq k$. It is clear that each $\Psi_{(k)} \in F$ and $\{\Psi_{(0)}, \ldots, \Psi_{(n-1)}\}$ is linearly independent in F . Thus this set is a basis for F .

Consider $\Psi^N \in F$. It can be uniquely written as $\Psi^N = c_0\Psi_{(0)} + \ldots + c_{n-1}\Psi_{(n-1)}$. Therefore we must only show that $\sum_{s=0}^{n-1} c_s = 1$ and $c = (c_0, \ldots, c_{n-1}) \geq 0$; the desired result will then follow upon taking $p^n_s = \binom{n-1}{s}^{-1} c_s$, yielding $\Psi^N = \Psi^N_{p^n}$. Suppose some $c_k < 0$. Consider $w \in G^N$ defined by $w(T) = 1$ if $|T| > k$, $w(T) = 0$ otherwise. Then for any $i \in N$, $(\Psi^N w)(i) = c_k(\Psi_{(k)}w)(i) = c_k < 0$; this contradicts the monotonicity axiom (3). Next consider $v_{\{1\}} \in G^N$.

By the projection axiom (4), we must have $(\Psi^N v_{\{1\}})(1) = \sum_{s=0}^{n-1} c_s = v_{\{1\}}(1) = 1$. □

Proof of Theorem 1a. It is straightforward to verify that each Ψ_ξ is a semivalue. Consider any semivalue Ψ . For each finite $N \subset U$, Ψ induces a semivalue Ψ^N on G^N . From the preceding lemma we know that each Ψ^N has the form

$$(\Psi^N v)(i) = \bigcup_{S \subset N \setminus i} p_s^N [v(S \cup i) - v(S)]$$

where all $p_s^N \geq 0$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^N = 1$. Furthermore, it is a simple consequence of the symmetry axiom that there is a collection of constants $\{p_s^n : s = 0, \ldots, n-1; n = 1, 2, \ldots\}$ such that for all $i \in N \subset U$ and $S \subset N \setminus i$, $p_s^N = p_s^n$.

Consider the collection of games $\{\hat{v}_S^N\}$, where $\hat{v}_S^N$ in G^N is defined for any $S \subset N \subset U$ by $\hat{v}_S^N(T) = 1$ if $T \supseteq S$, and 0 otherwise. For any $i \in N \setminus S$,

$$\Psi^N(\hat{v}_S^N)(i) = p_s^N = p_s^n .$$

For any given player $d \in U \setminus N$, the game $\hat{v}_S^N$ can be viewed as a game in $G^{N \cup d}$. It is easily shown that for any $i \in N \setminus S$,

$$\Psi^{N \cup d}(\hat{v}_S^N)(i) = p_s^{N \cup d} + p_s^{N \cup d} = p_s^{n+1} + p_{s+1}^{n+1}$$

Since Ψ^N and $\Psi^{N \cup d}$ are restrictions of the same operator Ψ , it follows that for any $i \in N \setminus S$,

$$\Psi^N(\hat{v}_S^N)(i) = p_s^n = p_s^{n+1} + p_{s+1}^{n+1} = \Psi^{N \cup d}(\hat{v}_S^N)(i) . \tag{1.3}$$

For notational ease, set $\alpha_n = p_n^{n+1}$ (for $n = 0, 1, 2, \ldots$). Obviously, p_s^n determines $\{\alpha_n\}_{n=0}^{\infty}$. Moreover, using (1.3) it can be shown by induction that for any $0 \leq s \leq n$,

$$p_s^{n+1} = (-1)^{n-s}\left(\alpha_n - \binom{n-s}{1}\alpha_{n-1} + \binom{n-s}{2}\alpha_{n-2} + \ldots + (-1)^{n-s}\alpha_s\right)$$

$$= (-1)^{n-s}\Delta^{n-s}\alpha_n ,$$

where Δ is the standard "backwards difference" operator. Consequently, we see that every sequence $\{\alpha_n\}$ of real numbers uniquely defines a collection $\{p_s^n\}$. It can be shown by direct summation that, for each n , the numbers $\left(\binom{n-1}{s}p_s^n\right)_{s=0}^{n-1}$ add to α_0 . Therefore, the collection $\{p_s^n\}$ will define a semivalue if and only if $\alpha_0 = 1$ and all $p_s^n \geq 0$.

It is well-known (for example, Theorem 4.6 of [3]) that a sequence $\{\alpha_n\}$ (with $\alpha_0 = 1$) and the successive differences $(-1)^k\Delta^k\alpha_n$ of all orders are nonnegative if and only if $\alpha_0, \alpha_1, \ldots$ are the moments of a uniquely-determined probability distribution ξ on $[0,1]$. In this case, since each $\alpha_n = \int_0^1 t^n d\xi(t)$, it follows that each

$$p_s^{n+1} = \int_0^1\left(t^s - \binom{n-s}{1}t^{s+1} + \ldots + (-1)^{n-s}t^n\right)d\xi(t)$$

$$= \int_0^1 t^s(1-t)^{n-s}d\xi(t) \ . \ \square$$

Alternative Proof of Theorem 1a.

It suffices to establish that Ψ is of the form Ψ_ξ for a unique probability measure ξ on $[0,1]$. Let $i \in U$ be fixed. For each finite subset N of $U \backslash i$, Ψ induces a semivalue on $G^{N \cup i}$, and hence, by Lemma 1, induces a probability measure c_N on the subsets of N such that $c_N(S) = p_s^{n+1}$. If $N \subset \overline{N}$, then by considering the natural embedding of G^N into $G^{\overline{N}}$, we have $c_N(S) = \sum c_{\overline{N}}(T)$, where the summation runs over all T for which $S \subset T \subset \overline{N}$ and $T \cap N = S$. Let $\{N_k\}$ be an increasing sequence of finite subsets of $U \backslash i$. The measures on the subsets of the various N_k are "consistent," and therefore by Kolmogorov's consistency theorem ([5], p. 94), there is a sequence of (0,1)-valued random variables $\{Y_j : j \in \cup N_k\}$ such that $c_{N_k}(S) = \text{Prob}(\{j : Y_j = 1\} = S)$. Thus $\{Y_j\}$ is an exchangeable sequence of random variables. De Finetti's theorem ([4], sec. 9.6.1) asserts that the distribution of every exchangeable infinite sequence of random variables is a unique mixture of distributions of sequences of independent identically-distributed random variables. As $\text{Prob}(Y_j = 0 \text{ or } 1) = 1$, there exists a unique probability measure ξ on $[0,1]$ such that for every finite sequence $\{\varepsilon_j : j \in N\}$ of 0's and 1's , $\text{Prob}(Y_j = \varepsilon_j \text{ for all } j \in N) = \int_0^1 t^{\Sigma\varepsilon_j}(1-t)^{n-\Sigma\varepsilon_j} d\xi(t)$ $= c_N(\{j : \varepsilon_j = 1\})$.

It is obvious from the axiom of symmetry that the mixing measure

depends neither on the particular player i , nor on the sequence N_k , and thus ξ is uniquely determined by Ψ alone. $\square$

This alternative proof provides another view of the theorem. Let $(\Omega,\mathcal{B},P)$ be a probability space, and $\{X_i : i \in U\}$ a family of independent identically-distributed random variable distributed uniformly on $[0,1]$. If $v \in G$ and $t \in [0,1]$, define the random variable $\Delta v(t)$ by $\Delta v(t) = v(\{i : X_i \leq t\}) - v(\{i : X_i < t\})$. We then have the following restatement of Theorem 1a:

Theorem 1a'. For each probability measure ξ on $[0,1]$ there is a semivalue Ψ_ξ on G defined by

$$(\Psi_\xi v)(i) = \int_0^1 E(\Delta v(t) | X_i = t) \cdot d\xi(t) .$$

Moreover, every semivalue on G is of this form and the mapping $\xi \to \Psi_\xi$ is 1-1 .

The Shapley value [8] is defined as $\phi = \Psi_\lambda$, where λ denotes the Lebesque measure on $[0,1]$. This is the only semivalue which has the efficiency property: for every $N \subset U$ and $v \in G^N$, $\phi v(N) = v(N)$. Define the bounded-variation norm of a game $v \in G$ with support N , as $\|v\| = \inf(v_+(N) + v_-(N))$, where the infimum is taken over all pairs v_+, v_- of monotonic games for which $v = v_+ - v_-$. With respect to this norm on G , the Shapley value is a continuous linear operator of norm 1. (For any monotonic $v_+, v_- \in G^N$ such that $v = v_+ - v_-$, $\|\phi v\| = \sum |\phi v(i)| \leq \sum(\phi v_+(i) + \phi v_-(i)) = v_+(N) + v_-(N)$; hence $\|\phi v\| \leq \|v\|$. But for any monotonic $v \in G^N$, $\|\phi v\| = v(N) = \|v\|$.)

We shall characterize the class of continuous semivalues on G .

Let W be the subset of $L_\infty(0,1)$ of all nonnegative functions g with $\int_0^1 g(t)dt = 1$.

<u>Theorem 1b</u>. For each $g \varepsilon W$, the operator $\Psi_g : G \to AG$ defined by

$$\Psi_g v(i) = \int_0^1 E(\Delta v(t) | X_i = t)\cdot g(t)dt$$

is a continuous semivalue. Moreover, every continuous semivalue on G is of this form. The map $g \to \Psi_g$ is a linear isometry (that is, $||\Psi_g|| = ||g||_{L_\infty}$) .

<u>Proof</u>. Consider any $g \varepsilon W$ and define $\xi = \int g d\lambda$. By Theorem 1a', $\Psi_g = \Psi$ is a semivalue. For any $v \varepsilon G^N$, and monotonic games v_+, v_- with $v = v_+ - v_-$, $||\Psi_g v|| = \sum|\Psi_g v(i)| \leq \sum|\Psi_g v_+(i)| + \sum|\Psi_g v_-(i)| \leq$ $||g||\cdot(\sum|\phi v_+(i)| + \sum|\phi v_-(i)|) = ||g||\cdot(v_+(N) + v_-(N))$; therefore, $||\Psi_g v|| \leq ||g||\cdot||v||$. Hence Ψ_g is continuous, and $||\Psi_g|| \leq ||g||$.

Next, consider any continuous semivalue Ψ_ξ . Select any (relatively) open interval $J \subset [0,1]$, and assume that $\xi(J) = M\cdot\lambda(J)$. Fix a player $i \varepsilon U$, and for each $k > 0$, select $N_k \subset U$ such that $i \varepsilon N_k$ and $|N_k| = k$. Let $v_k \varepsilon G^{N_k}$ be defined by $v_k(S) = \lambda([0, \frac{s}{n}] \cap J)$. By the law of large numbers, $\liminf \Psi_\xi v_k(i) \geq \frac{1}{n}\cdot\xi(J) = \frac{M}{n}\cdot\lambda(J)$. Therefore $||\Psi_\xi v_k|| = \sum|\Psi_\xi v_k(i)| \geq M\cdot\lambda(J)$, while each $||v_k|| = \lambda(J)$. Hence, $||\Psi_\xi|| \geq M$. The continuity of Ψ_ξ implies that $||\Psi_\xi||$ is finite. Consequently, $\overline{M} = \sup\{\xi(J)/\lambda(J) : J$ is an interval in $[0,1]\} < \infty$, and the Radon-Nikodym derivative $d\xi/d\lambda = g$ is in W . Therefore $\Psi_\xi = \Psi_g$, and $||\Psi_g|| \geq \overline{M} = ||g||$. $\square$

2. The Infinite Case

All definitions and notation are according to [1]. Let $(I,\mathcal{C})$ be a measure space isomorphic to $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of $[0,1]$. The members of I are called players, the members of $\mathcal{C}$ coalitions, and set functions are called games. Let BV be the space of bounded-variation set functions on $(I,\mathcal{C})$. The space of all bounded, finitely-additive set functions is denoted FA , and its subspace of all nonatomic measures is denoted NA . Denote by $\mathcal{G}$ the group of automorphisms of $(I,\mathcal{C})$. For each $\theta \in \mathcal{G}$, $\theta^* : BV \to BV$ is defined by $\theta^* v(S) = v(\theta S)$. If $Q \subset BV$ then Q^+ denotes the subset of Q of all monotonic set functions. A subset Q of BV is symmetric if for each $\theta \in \mathcal{G}$, $\theta^* Q \subseteq Q$. An operator $\Psi : Q \to BV$ is called positive if $\Psi(Q^+) \subset BV^+$, and symmetric if for each $\theta \in \mathcal{G}$, $\theta^*\Psi = \Psi\theta^*$.

Let Q be a linear symmetric subspace of BV . A semivalue on Q is an operator Ψ from Q into FA such that:

(1) Ψ is linear,

(2) Ψ is symmetric,

(3) Ψ is positive,

(4) if $v \in Q \cap FA$ then $\Psi v = v$.

We will characterize the semivalues on pNA , the closed subspace of BV spanned by all powers of NA^+ measures. This space plays an important role in the theory of nonatomic games, and contains many games of interest. For example, pNA contains all "vector measure games" satisfying appropriate differentiability conditions, i.e., all set functions of the form $f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a nonatomic finite-dimensional vector measure and f is an appropriately differentiable real-valued function defined on the range of μ , with $f(0) = 0$. As

our main theorem in this section uses notation and terminology related to the "extension" of a game, we restate here relevant definitions and results from [1]. $\mathcal{I}$ denotes the family of all measurable functions from $(I,\mathcal{C})$ to $([0,1], \mathcal{B})$. There is a partial order on $\mathcal{I}$: $f \geq g$ if $f(s) \geq g(s)$ for all $s \in I$. A real valued function w on $\mathcal{I}$ with $w(0) = 0$ is called an <u>ideal set function</u>; it is called <u>monotonic</u> if $f \geq g$ implies $w(f) \geq w(g)$. The characteristic function of a member S of $\mathcal{C}$ is denoted χ_S. We will sometimes denote χ_S by S and $t\cdot\chi_I$ by t.

It is shown in [1; Theorem G] that there is a unique monotonicity-preserving linear mapping which associates with each $v \in pNA$ an ideal set function v^*, such that $(v\cdot w)^* = v^*\cdot w^*$ for all $v, w \in pNA$, and $\mu^*(f) = \int_I f\cdot d\mu$ for all $\mu \in NA$ and $f \in \mathcal{I}$.

Denote $\partial v^*(t,S) = (d/d\tau)\cdot v^*(t\chi_I + \tau\cdot\chi_S)\big|_{\tau=0}$. By Theorem H of [1] we know that for each $v \in pNA$ and each $S \in \mathcal{C}$, the derivative $\partial v^*(t,S)$ exists for almost all t in $[0,1]$, and is integrable over $[0,1]$ as a function of t.

Recall that W is the set of nonnegative functions $g \in L_\infty(0,1)$ such that $\int_0^1 g(t)dt = 1$.

<u>Theorem 2</u>. For each $g \in W$ the operator $\Psi_g : pNA \to FA$ defined by

$$\Psi_g v(S) = \int_0^1 \partial v^*(t,S)\cdot g(t)dt$$

is a semivalue. Moreover, every semivalue on pNA is of this form. The map $g \to \Psi_g$ of W onto the family of semivalues on pNA is a linear isometry.

Proof. Let $g \in W$ be given. For $v \in pNA$, Lemma 23.1 of [1] asserts that $\int_0^1 |\partial v^*(t,S)|\cdot dt \leq \|v\|$. Hence $|\Psi_g v(S)| = |\int_0^1 \partial v^*(t,S)\cdot g(t)dt| \leq \|g\|\cdot\|v\|$; this proves that $\Psi_g v$ is bounded. If $S, T \subset I$ with $S \cap T = \emptyset$ then $\partial v^*(t, T \cup S) = \partial v^*(t,T) + \partial v^*(t,S)$ for almost all t. Therefore $\Psi_g v(S \cup T) = \Psi_g v(S) + \Psi_g(T)$, which proves that Ψ_g takes pNA into FA. Linearity of Ψ_g follows from the linearity of the extension as well as that of the derivative. Symmetry of Ψ_g follows from the fact that $\partial(\theta^*v)^*(t,S) = \partial v^*(t,\theta S)$ and thus $\theta^*\Psi_g v(S) = \int\partial v^*(t,\theta S)\cdot g(t)dt$ $= \int\partial(\theta^*v)^*(t,S)\cdot g(t)dt = \Psi_g(\theta^*v)(S)$. Let $v \in pNA^+$. Then v^* is also monotonic and $\partial v^*(t,S) \geq 0$; thus $\Psi_g v$ is monotonic, which proves the positivity of Ψ_g. Finally, any $u \in pNA \cap FA$ is in NA (Corollary 5.3 of [1], and the continuity of the elements of the space AC ([1], page 205), imply that u is countably additive). Hence $\partial u^*(t,S) = u(S)$ and consequently $\Psi_g u = u$. This completes the proof that Ψ_g is a semi-value.

Now, let Ψ be a semivalue on pNA. Let μ be a fixed probability measure in NA. Each $f \in L_1$ induces a game v_f defined by

$$v_f(S) = \int_0^{\mu(S)} f(t)dt .$$

In other words, f defines a function $F : [0,1] \to R$ by $F(s) = \int_0^s f(t)dt$, and $v_f = F\circ\mu$. As $f \in L_1$, F is absolutely continuous and therefore $v_f \in pNA$. In analogy with the proof of Proposition 6.1 of [1] it follows that $\Psi v_f = C(f)\cdot\mu$, where $C(f)$ is a constant independent of μ. Observe that $v_{f+g} = v_f + v_g$; thus the linearity of Ψ implies that C is linear. We now proceed to show that C is continuous. Observe that

$\|v_f\| = \|f\|_{L_1}$. Since pNA is internal ([1], Proposition 7.19), it is a closed reproducing space and thus ([1], Proposition 4.3) Ψ is continuous on pNA . That is, there exists a constant K with $\|\Psi v\| \leq K \cdot \|v\|$, which in particular implies that $|C(f)| = \|C(f) \cdot \mu\| \leq K \cdot \|v_f\| = K \cdot \|f\|_{L_1}$. Hence $C : L_1 \to R$ is a continuous linear functional and therefore is of the form $C(f) = \int_0^1 f(t)\, g(t)dt$ for some $g \in L_\infty$. We shall show that $\Psi = \Psi_g$. As was shown in the beginning of the proof, $\Psi_g(pNA) \subset FA$ and $|\Psi_g v(S)| \leq \|g\| \cdot \|v\|$, which implies that Ψ_g is continuous. For each $f \in L_1$, $\partial v_f^*(t,S) = f(t) \cdot \mu(S)$ for almost all t , and thus $\Psi_g v_f(S) = \mu(S) \cdot \int f(t)g(t)dt = C(f) \cdot \mu(S) = \Psi v_f(S)$ and therefore $\Psi_g v_f = \Psi v_f$. The linear symmetric subspace spanned by $\{v_f : f \in L_1\}$ is dense in pNA (it contains all powers of NA measures). The operators Ψ and Ψ_g are linear and symmetric and thus coincide on this subspace; as they are also continuous, they coincide on pNA . It remains for us to show that $g \in W$. For $v \in NA \subset FA \cap pNA$, it follows that $\partial v^*(t,S) = v(S)$. Thus $\Psi_g v(S) = (\int_0^1 g(t)dt)v(S)$, which shows that $\int_0^1 g(t)dt = 1$. Let $B_\varepsilon = \{t : g(t) \leq -\varepsilon\}$ and let f be the characteristic function of B_ε . Then $f \geq 0$ and hence v_f is monotonic. But as $\Psi_g v_f(I) = \int f(t)g(t)dt \leq -\varepsilon \cdot \lambda(B_\varepsilon)$ (λ denotes the Lebesgue measure on $[0,1]$) and $\Psi_g = \Psi$ is positive, it must be that $\lambda(B_\varepsilon) = 0$. As this holds for any $\varepsilon > 0$, g is nonnegative. This completes the proof that any semivalue Ψ is of the form Ψ_g for some $g \in W$.

Now, for any $g \in W$ and $\varepsilon > 0$ there exists a nonnegative $f \in L_1$ with $\|f\|_{L_1} = 1$ and $\int f(t)g(t)dt = \|g\| - \varepsilon$. Observe that $\|v_f\| = \|f\|_{L_1} = 1$ and that $\|\Psi_g v_f\| = \|g\| - \varepsilon$; hence $\|\Psi_g\| \geq \|g\|$. On the other hand, for $v \in pNA^+$,

$$\|\Psi_g v\| = \Psi_g v(I) = \int \partial v^*(t,I)\cdot g(t)dt \le \|g\|\cdot\int_0^1 \partial v^*(t,I)\cdot dt = \|g\|\cdot\|v\| .$$

In the general case, when v is not necessarily monotonic, let $\varepsilon > 0$ be given. Set $v = u-w$, where u and w are in pNA^+ and $\|v\| + \varepsilon \geq \|u\| + \|w\|$; such u and w exist because pNA is internal. Then $\|\Psi_g v\| \le \|\Psi_g u\| + \|\Psi_g w\| \le \|g\|(\|u\| + \|w\|) \le \|g\|(\|v\| + \varepsilon)$, and if we let $\varepsilon \to 0$, $\|\Psi_g v\| \le \|g\|\cdot\|v\|$; this completes the proof of the equality $\|\Psi_g\| = \|g\|$. $\square$

3. Remarks

Continuous semivalues are diagonal. (The proof in [6] that continuous values are diagonal does not make use of the efficiency axiom and therefore the same proof works here.) Furthermore, semivalues on closed reproducing spaces are diagonal.

The semivalues derived axiomatically on pNA can also be obtained from a complementary, asymptotic point of view [2] which links the finite-player and nonatomic approaches.

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